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Explicit Construction of Multivariate Padé Approximants for a q-Logarithm Function

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We explicitly construct the non-homogeneous multivariate Padé approximants to a two variable version of the q-logarithm function

$$L_q(x, y) := \sum_{i, j=0}^{\infty} \frac{(q-1)\, x^i y^j}{q^{i+j+1}-1},$$

for $|q| > 1$ and $|x|, |y| < |q|$, by using the residue theorem and functional equation method. © 2000 Academic Press

Key Words: multivariate Padé approximants; q-logarithm functions; functional equations.

1. INTRODUCTION

By using the residue theorem and the functional equation method, Borwein [1] explicitly constructed the Padé approximants to a q analogue of the one variable exponential, logarithm and partial theta functions. By using the similar but more technique method, the author explicitly constructed in [10] and [11] the general multivariate Padé approximants to some q-functions which satisfy some functional equations, e.g.

$$F_q(x, y) := \prod_{j=0}^{\infty} (1 + q^{-j}x + q^{-2j}xy), \qquad |q| > 1, \tag{1.1}$$

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a two variable version of a infinite product discussed by Bundschuh [2], Wallisser [9] and many other researchers; and

$$E_q(x, y) := \sum_{i, j=0}^{\infty} \frac{x^i y^j}{[i+j]_q!}, \qquad |q| > 1, \tag{1.2}$$

a two variable version of the q-exponential function discussed by Borwein [1], Mahler [8], Wallisser [9] and many others (See (1.12) below for the notation of $[n]_q!$ for integers n.); and

$$T_q(x, y) := \sum_{i, j=0}^{\infty} q^{(i+j)(i+j-1)/2} x^i y^j, \qquad |q| < 1, \tag{1.3}$$

a two variable version of the partial theta function discussed by Borwein [1], Lubinsky and Saff [7] and many others. By constructing the similar approximants to the function $F_q(x, y)$ defined by (1.1), the author proved in [12] that if q is an integer greater than one, r and s are any rationals, then

$$\prod_{j=0}^{\infty} (1 + q^{-j} r + q^{-2j} s)$$

is irrational.

As the situation is complicated when we deal with multivariate functions, explicit construction of multivariate Padé approximants to a function is challenging. Our intention in this paper is to show how to construct the general multivariate Padé approximants to a two variable version of the q-logarithm. In order to avoid notational difficulties to introduce the definition of multivariate Padé approximants, we restrict ourselves to the case of bivariate functions. The generalization to more than two variables is straightforward (see Cuyt [5]).

DEFINITION. Let

$$F(x, y) := \sum_{(i, j) \in \mathbb{N}^2} c_{ij} x^i y^j, \qquad c_{ij} \in \mathbb{C} \tag{1.4}$$

be a formal power series, and let M, N, E be index sets in $\mathbb{N} \times \mathbb{N} =: \mathbb{N}^2$. The (M, N) *general multivariate Padé' approximant to* $F(x, y)$ *on the finite set* E is a rational function

$$[M/N]_E(x, y) := \frac{P(x, y)}{Q(x, y)} \tag{1.5}$$

with polynomials

$$P(x, y) := \sum_{(i, j) \in M} a_{ij} x^i y^j, \qquad a_{ij} \in \mathbb{C}, \tag{1.6}$$

$$Q(x, y) := \sum_{(i, j) \in N} b_{ij} x^i y^j, \qquad b_{ij} \in \mathbb{C}, \tag{1.7}$$

and interpolation set E, such that

$$(FQ - P)(x, y) = \sum_{(i, j) \in \mathbb{N}^2 \backslash E} d_{ij} x^i y^j, \qquad d_{ij} \in \mathbb{C} \tag{1.8}$$

with

$$M \subseteq E, \tag{1.9}$$

$$\#(E \backslash M) \geqslant \# N - 1 \tag{1.10}$$

and E satisfies the inclusion property:

$$(i, j) \in E, \qquad 0 \leqslant k \leqslant i, \qquad 0 \leqslant l \leqslant j \Rightarrow (k, l) \in E. \tag{1.11}$$

One may find properties of general multivariate Padé approximants discussed in Cuyt [3], [4]. We need here the standard q analogues of factorials and binomial coefficients. The q-factorial is

$$[n]_q! := [n]! := \frac{(1-q^n)(1-q^{n-1}) \cdots (1-q)}{(1-q)^n}, \tag{1.12}$$

where $[0]_q! := 1$. The *q-binomial coefficient* is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \begin{bmatrix} n \\ k \end{bmatrix} := \frac{[n]!}{[k]! \cdot [n-k]!}. \tag{1.13}$$

Note that

$$[n]_{q^{-1}}! = q^{-n(n-1)/2} [n]!, \tag{1.14}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}, \tag{1.15}$$

$$\prod_{\substack{h=0 \\ h \neq k}}^{n} (q^{-k} - q^{-h}) = (-1)^k \, q^{-k(k-1)/2 - n(n+1)/2} [n]! \, [k]! \, (1-q)^n, \tag{1.16}$$

and (see Gasper and Rahman [6]) for $|t| < q^{-n}$,

$$\frac{1}{\prod_{k=0}^{n} (t - q^{-k})} = (-1)^{n+1} q^{n(n+1)/2} \sum_{l=0}^{\infty} \begin{bmatrix} n+l \\ l \end{bmatrix} t^l. \tag{1.17}$$

We also need the Cauchy binomial theorem

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2} x^k = \prod_{k=1}^{n} (1 + q^k x). \tag{1.18}$$

We state our results in Section 2 and proof the results in Section 3.

2. EXPLICIT CONSTRUCTION OF THE APPROXIMANTS

The functional equation a function satisfies and the appropriate integral we construct play the crucial roles in finding the explicit formulae for multivariate Padé approximants to this function. The functional equation used in this paper is simple while the integral is relatively complicated.

Let $|q| > 1$, $|x|$, $|y| < q$,

$$L(x, y) := L_q(x, y) := \sum_{i, j=0}^{\infty} \frac{(q-1)\, x^i y^j}{q^{i+j+1} - 1}. \tag{2.1}$$

As

$$q^{i+j+1} - 1 = (q-1)(q^{i+j} + q^{i+j-1} + \cdots + 1),$$

we have

$$\lim_{q \to 1} L(x, y) = \sum_{i, j=0}^{\infty} \frac{x^i y^j}{i+j+1}$$
$$= \frac{1}{y-x} [\ln(1-x) - \ln(1-y)]. \tag{2.2}$$

So we say that $L(x, y)$ is a two variable q-analogue of the logarithm function. Now for $k \geqslant 0$ integer, and $|x|$, $|y| < |q|$,

$$\begin{aligned}L(q^{-1}x, q^{-1}y) &= \sum_{i,\,j=0}^{\infty} \frac{(q-1)\,q^{-(i+j)}x^i y^j}{q^{i+j+1}-1} \\ &= \sum_{i,\,j=0}^{\infty} \frac{(q-1)(1-q^{i+j+1}+q^{i+j+1})\,x^i y^j}{q^{i+j}(q^{i+j+1}-1)} \\ &= \sum_{i,\,j=0}^{\infty} \frac{q(q-1)\,x^i y^j}{q^{i+j+1}-1} - (q-1) \sum_{i,\,j=0}^{\infty} \frac{x^i y^j}{q^{i+j}} \\ &= qL(x, y) - \frac{(q-1)}{(1-q^{-1}x)(1-q^{-1}y)},\end{aligned}$$

so

$$\begin{aligned}L(q^{-k}x, q^{-k}y) &= q^k L(x, y) - \sum_{j=1}^{k} \frac{(q-1)\,q^{k-j}}{(1-q^{-j}x)(1-q^{-j}y)} \\ &=: q^k L(x, y) - S_k(x, y),\end{aligned} \tag{2.3}$$

where

$$S_k(x, y) := \sum_{j=1}^{k} \frac{(q-1)\,q^{k-j}}{(1-q^{-j}x)(1-q^{-j}y)}. \tag{2.4}$$

THEOREM. *Let $L(x, y)$ and $S_k(x, y)$ be defined by (2.1) and (2.4), and*

$$R_n(x, y) := \prod_{j=0}^{n-1} (1-q^j x)(1-q^j y). \tag{2.5}$$

Let $m, n \in \mathbb{N}$, $m \geqslant n+1 \geqslant 1$, and

$$W := \{(i, j) \colon 0 \leqslant i+j \leqslant m,\ i, j \geqslant 0\}, \tag{2.6}$$

$$N := \{(i, j) \colon 0 \leqslant i, j \leqslant n\}, \tag{2.7}$$

$$M := N \cup W, \tag{2.8}$$

$$E := \{(i, j) \colon 0 \leqslant i+j \leqslant m+n,\ i, j \geqslant 0\}. \tag{2.9}$$

Let

$$I(x, y) := \frac{1}{2\pi i} \int_{\Gamma} \frac{R_n(tx, ty)\, L(tx, ty)\, dt}{\left(\prod_{k=0}^{n} (t-q^{-k})\right) t^{m+1}}, \tag{2.10}$$

where Γ is a circular contour containing $0, q^0, q^{-1}, ..., q^{-n}$, and

$$Q(x, y) := \frac{q^{n(n+1)/2}}{(1-q)^n\,[n]!} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{mk+k(k+3)/2} R_n(q^{-k}x, q^{-k}y), \tag{2.11}$$

and

$$\begin{aligned} P(x, y) := {} & \frac{q^{n(n+1)/2}}{(1-q)^n\,[n]!} \sum_{k=0}^{n} (-1)^{k+1} \\ & \times \begin{bmatrix} n \\ k \end{bmatrix} q^{mk+k(k+3)/2} R_n(q^{-k}x, q^{-k}y)\, S_k(x, y) \\ & + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{R_n(tx, ty)\, F(tx, ty)}{\prod_{k=0}^{n} (t-q^{-k})} \right\}_{t=0}. \end{aligned} \tag{2.12}$$

Then

(i) $$I(x, y) = Q(x, y)\, L(x, y) + P(x, y); \tag{2.13}$$

(ii) $$Q(x, y) = \sum_{(i, j) \in N} a_{ij} x^i y^j, \qquad a_{ij} \in \mathbb{C}, \tag{2.14}$$

$$P(x, y) = \sum_{(i, j) \in M} b_{ij} x^i y^j, \qquad b_{ij} \in \mathbb{C}; \tag{2.15}$$

(iii) $$I(x, y) = \sum_{(i, j) \in \mathbb{N}^2 \setminus E} d_{ij} x^i y^j, \qquad d_{ij} \in \mathbb{C}; \tag{2.16}$$

(iv) $$M \subseteq E, \qquad \text{and} \qquad \#(E \setminus M) \geqslant \#N - 1, \tag{2.17}$$

and then the (M, N) non-homogeneous multivariate Padé approximant to $L(x, y)$ on the set E is

$$[M/N]_E\,(x, y) = -\frac{P(x, y)}{Q(x, y)}.$$

Remark. For simplicity we focus on the two variable case above. In fact this result can be generalized to multivariable series

$$L_q(\underline{z}) := \sum_{j_1, ..., j_k = 0}^{\infty} \frac{z_1^{j_1} \cdots z_k^{j_k}}{q^{j_1 + \cdots + j_k + 1} - 1}, \tag{2.18}$$

where $\underline{z} := (z_1, ..., z_k) \in \mathbb{C}^k$, and $|z_l| < |q|$, $l = 1, ..., k$, by substituting (x, y) by $(z_1, ..., z_k)$ and $x^i y^j$ by $z_1^{j_1} \cdots z_k^{j_k}$, and W, N, M, E by

$$W^* := \{(j_1, ..., j_k) \in \mathbb{N}^k : 0 \leqslant j_1 + \cdots j_k \leqslant m\}, \tag{2.19}$$

$$N^* := \{(j_1, ..., j_k) \in \mathbb{N}^k : 0 \leqslant j_1, ..., j_k \leqslant n\}, \tag{2.20}$$

$$M^* := N \cup W, \tag{2.21}$$

$$E^* := \{(j_1, ..., j_k) \in \mathbb{N}^k : 0 \leqslant j_1 + \cdots + j_k \leqslant m + n\}, \tag{2.22}$$

respectively.

3. PROOF OF THE THEOREM

Proof of (i). We can see that the integrand in (2.10) has simple poles at $t = 1, q^{-1}, ..., q^{-n}$, and a pole of order $m + 1$ at $t = 0$, inside the contour Γ. By the residue theorem and the functional equation (2.3), and (1.13) to (1.16), we have

$$\begin{aligned} I(x, y) &= \frac{1}{2\pi i} \int_\Gamma \frac{R_n(tx, ty)\, L(tx, ty)\, dt}{(\prod_{k=0}^n (t - q^{-k}))\, t^{m+1}} \\ &= \sum_{k=0}^n \frac{R_n(q^{-k}x, q^{-k}y)\, L(q^{-k}x, q^{-k}y)}{(\prod_{h=0,\, h\neq k}^n (q^{-k} - q^{-h}))\, q^{-k(m+1)}} \\ &\quad + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{R_n(tx, ty)\, L(tx, ty)}{\prod_{k=0}^n (t - q^{-k})} \right\}_{t=0} \\ &= \frac{q^{n(n+1)/2}}{(1-q)^n\, [n]!} \sum_{k=0}^n (-1)^k \\ &\quad \times \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k+1)/2 + km} R_n(q^{-k}x, q^{-k}y)(q^k L(x, y) - S_k(x, y)) \\ &\quad + \frac{1}{m!} \frac{d^m}{dt^m} \left\{ \frac{R_n(tx, ty)\, L(tx, ty)}{\prod_{k=0}^n (t - q^{-k})} \right\}_{t=0} \\ &= Q(x, y)\, L(x, y) + P(x, y). \end{aligned}$$

Proof of (ii). It is easy to see from the definition of $Q(x, y)$ and $R_n(x, y)$ that (2.14) holds. Now for $0 \leqslant k \leqslant n$,

$$R_n(q^{-k}x, q^{-k}y) = \prod_{j=0}^{n-1} (1-q^{j-k}x)(1-q^{j-k}y)$$
$$= \left(\prod_{j=1}^{k} (1-q^{-j}x)(1-q^{-j}y)\right)\left(\prod_{j=0}^{n-k-1} (1-q^{j}x)(q^{j}y)\right),$$

then

$$R_n(q^{-k}x, q^{-k}y)\, S_k(x, y) = (q-1)\left(\prod_{j=0}^{n-k-1} (1-q^{j}x)(1-q^{j}y)\right)$$
$$\times \sum_{h=1}^{k} q^{k-h} \prod_{\substack{j=1 \\ j\neq h}}^{k} (1-q^{-j}x)(1-q^{-j}y), \tag{3.1}$$

and then

$$R_n(q^{-k}x, q^{-k}y)\, S_k(x, y) = \sum_{(i,\, j)\in N} s_{ij} x^i y^j, \qquad s_{ij} \in \mathbb{C}. \tag{3.2}$$

From the Cauchy binomial theorem (1.18),

$$R_n(tx, ty) = \prod_{j=0}^{n-1} ((1-q^{j}xt)(1-q^{j}yt))$$
$$= \left(\sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{k(k-1)/2} x^k t^k\right)\left(\sum_{l=0}^{n} (-1)^l \begin{bmatrix} n \\ l \end{bmatrix} q^{l(l-1)/2} y^l t^l\right)$$
$$= \sum_{k,\, l=0}^{n} (-1)^{k+l} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix} q^{k(k-1)/2 + l(l-1)/2} x^k y^l t^{k+l}. \tag{3.3}$$

Also

$$L(tx, ty) = \sum_{i,\, j=0}^{\infty} \frac{(q-1)\, x^i y^j t^{i+j}}{q^{i+j+1}-1}. \tag{3.4}$$

Then from (1.17), (3.3) and (3.4), for $|t| \leqslant q^{-n}$,

$$\frac{R_n(tx, ty)\, L(tx, ty)}{\prod_{k=0}^{n} (t-q^{-k})}$$
$$= (-1)^{n+1} q^{n(n+1)/2} \sum_{i,\, j,\, h=0}^{\infty} \sum_{k,\, l=0}^{n} (-1)^{k+l} \begin{bmatrix} n \\ k \end{bmatrix}\begin{bmatrix} n \\ l \end{bmatrix}\begin{bmatrix} n+h \\ h \end{bmatrix}$$
$$\times q^{k(k-1)/2 + l(l-1)/2} \frac{(q-1)\, x^{i+k} y^{j+l} t^{i+j+h+k+l}}{(q^{i+j+1}-1)}.$$

So

$$\frac{1}{m!}\frac{d^m}{dt^m}\left\{\frac{R_n(tx,ty)\,L(tx,ty)}{\prod_{k=0}^{n}(t-q^{-k})}\right\}_{t=0} = (-1)^{n+1}(q-1)\,q^{n(n+1)/2} \times \sum_{\substack{i+j+h+l+k=m\\0\leqslant i,j,h\leqslant m,\,0\leqslant k,l\leqslant n}} \frac{(-1)^{k+l}\,x^{i+k}y^{j+l}}{(q^{i+j+1}-1)} \times \begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n\\ l\end{bmatrix}\begin{bmatrix} n+h\\ h\end{bmatrix} q^{k(k-1)/2+l(l-1)/2}, \tag{3.5}$$

and then

$$\frac{1}{m!}\frac{d^m}{dt^m}\left\{\frac{R_n(tx,ty)\,L(tx,ty)}{\prod_{k=1}^{n}(t-q^{k})}\right\}_{t=0} = \sum_{(i,j)\in W} r_{ij}x^iy^j, \qquad r_{ij}\in\mathbb{C}. \tag{3.6}$$

Thus (2.15) follows from (3.2) and (3.6).

Proof of (iii). From (2.10), (3.3) and (3.4),

$$\begin{aligned}
I(x,y) &= \frac{1}{2\pi i}\int_{\Gamma}\frac{R_n(tx,ty)\,L(tx,ty)\,dt}{t^{m+n+2}(\prod_{k=0}^{n}(1-1/(q^kt))}\\
&= \frac{1}{2\pi i}\int_{\Gamma}\frac{R_n(tx,ty)\,L(tx,ty)}{t^{m+n+2}}\left(\sum_{j_0,\ldots,j_n\geqslant 0}\prod_{k=0}^{n}\left(\frac{1}{q^kt}\right)^{j_k}\right)dt\\
&= \sum_{j_0,\ldots,j_n\geqslant 0} q^{-\Sigma_{k=0}^{n}kj_k}\cdot\frac{1}{2\pi i}\int_{\Gamma}\left\{\frac{1}{t^{m+n+2+(j_0+\cdots+j_n)}}\right.\\
&\quad\times\sum_{i,j=0}^{\infty}\sum_{k,l=0}^{n}(-1)^{k+l}\begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n\\ l\end{bmatrix}q^{k(k-1)/2+l(l-1)/2}\\
&\quad\times\left.\frac{(q-1)\,x^{i+k}y^{j+l}t^{i+j+k+l}}{(q^{i+j+1}-1)}\right\}dt\\
&= \sum_{j_0,\ldots,j_n\geqslant 0} q^{-\Sigma_{k=0}^{n}kj_k}\sum_{\substack{i+j+l+k-(m+n+j_0+\cdots j_n+2)=-1\\0\leqslant i,j\leqslant\infty,\,0\leqslant l,k\leqslant n}}(-1)^{k+l}\begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n\\ l\end{bmatrix}\\
&\quad\times q^{k(k-1)/2+l(l-1)/2}\frac{(q-1)\,x^{i+k}y^{j+l}}{(q^{i+j+1}-1)}\\
&= \sum_{\substack{i+j+l+k=m+n+j_0+\cdots j_0+1\\0\leqslant i,j\leqslant\infty,\,0\leqslant l,k\leqslant n\\j_0+\cdots+j_n\geqslant 0}} q^{-\Sigma_{k=0}^{n}kj_k}(-1)^{k+l}\begin{bmatrix} n\\ k\end{bmatrix}\begin{bmatrix} n\\ l\end{bmatrix}\\
&\quad\times q^{k(k-1)/2+l(l-1)/2}\frac{(q-1)\,x^{i+k}y^{j+l}}{(q^{i+j+1}-1)}.
\end{aligned}\tag{3.7}$$

So (2.16) holds.

Proof of (iv). $M \subseteq E$ is obvious, and

$$\# W = \# \{(i, j)\colon 0 \leqslant i + j \leqslant m,\ i, j \geqslant 0\} = \frac{(m+1)(m+2)}{2}.$$

For $n < m < 2n$, i.e. $n + 1 \leqslant m \leqslant 2n - 1$,

$$m - n \geqslant 1, \qquad \text{and} \qquad 2n - m \geqslant 1, \tag{3.8}$$

and

$$\begin{aligned} \# M &= \# N + 2 \cdot \frac{(m-n)(m-n+1)}{2} \\ &= (n+1)^2 + (m-n)(m-n+1) \\ &= m^2 + 2n^2 - 2mn + m + n + 1, \\ \# E &= \frac{(m+n+1)(m+n+2)}{2}, \end{aligned}$$

so

$$\begin{aligned} \# (E \backslash M) &= \tfrac{1}{2}(m+n+1)(m+n+2) - (m^2 + 2n^2 - 2mn + m + n + 1) \\ &= 3mn - \tfrac{1}{2}m^2 - \tfrac{3}{2}n^2 + \tfrac{1}{2}m + \tfrac{1}{2}n \\ &= mn - \tfrac{1}{2}m(m-n) + \tfrac{3}{2}n(m-n) + \tfrac{1}{2}(m+n) \\ &= mn + \tfrac{1}{2}(m-n)(3n-m) + \tfrac{1}{2}(m+n) \\ &\geqslant mn + \tfrac{1}{2}(n+1) + \tfrac{1}{2}(m+n) \qquad (\text{by } (3.8)) \\ &\geqslant (n+1)\, n + \tfrac{1}{2}n + \tfrac{1}{2}(n+n) + \tfrac{1}{2} \\ &\geqslant n^2 + 2n \\ &= \# N - 1. \end{aligned}$$

Now for $m \geqslant 2n$, we have $N \subseteq M$, and

$$E \backslash M = \{(i, j)\colon m + 1 \leqslant i + j \leqslant m + n,\ i, j \geqslant 0\},$$

then

$$\begin{aligned} \# (E \backslash M) &= \frac{(m+n+1)(m+n+2)}{2} - \frac{(m+1)(m+2)}{2} \\ &= \frac{n(2m+n+3)}{2} \\ &\geqslant \frac{n(5n+3)}{2} \qquad (\text{as } m \geqslant 2n) \\ &\geqslant n^2 + 2n \\ &= \# N - 1. \end{aligned} \tag{3.9}$$

Then for all $m \geqslant n+1$,

$$\#(E \backslash M) \geqslant \#N - 1.$$

Combining (i), (ii), (iii) and (iv), we have

$$[M/N]_E(x, y) = -\frac{P(x, y)}{Q(x, y)}.$$

This completes the proof of the Theorem.

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